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Remarks on the Integrability of First-Order Complex PDE

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The classical method of characteristics for the integration of real first-order PDE is extended to complex equations, via a symplectic structure on the complex cotangent bundle that is holomorphic along the fibres. Integrability is shown to be equivalent to a variant of Darboux's theorem, i.e., the existence of symplectic "frames" that incorporate the equations. Applications of the linear theory to the associated Hamiltonian system yield integrability results for the nonlinear PDE.

CONTENTS

INTRODUCTION

The purpose of the present article is to clarify the relation between the solvability of an overdetermined system of first-order nonlinear PDE,

$$p_j(x, dw) = 0, \qquad j = 1, ..., n, \tag{1}$$

and the integrability of the associated Hamiltonian system,

$$\mathbf{H}_{p_j} = \sum_{k=1}^{N} (\partial p_j/\partial \xi_k)\, \partial/\partial x_k - (\partial p_j/\partial x_k)\, \partial/\partial \xi_k, \qquad j = 1, ..., n. \tag{2}$$

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Equations (1) are complex and, in general, the solution w will not be real. One must therefore hypothesize that the functions $p_j(x, \xi)$ are holomorphic with respect to the "fibre variables" ξ_k, which vary in $\mathbb{C}$.

The article extends to this stuation the classical method of characteristics. However, since the base manifold $\mathcal{M}$ in which x varies is real, and the functions p_j are merely $\mathscr{C}^\infty$, not analytic, in general characteristics will not exist. Such "facts of life" steer the course of generalization. The symplectic geometry foundation must be adapted to a set-up in which part of the variables (x) are real and part (ξ) are complex, and all functions in the complexified cotangent bundle $\mathbb{C}\mathbf{T}^*\mathcal{M}$ are holomorphic along the fibres $\mathbb{C}\mathbf{T}^*_x\mathcal{M}$. The (complex) symplectic structure is not carried by the complexified tangent spaces to $\mathbb{C}\mathbf{T}^*\mathcal{M}$ but by their quotients modulo the space of vectors tangent to the fibres that are of type (0, 1), i.e., modulo the linear combinations of the Cauchy Riemann operators $\partial/\partial\bar{\xi}_i$. The sections of the resulting vector bundle are not true vector fields; but they act as true vector fields on functions $f(x, \xi)$ that are holomorphic with respect to ξ. This is how the Hamiltonian fields (2) must be interpreted.

This symplectic geometry, half real and half complex as it is, calls into play only submanifolds of $\mathbb{C}\mathbf{T}^*\mathcal{M}$ whose intersections with any fibre $\mathbb{C}\mathbf{T}^*_x\mathcal{M}$ are complex analytic submanifolds of $\mathbb{C}\mathbf{T}^*_x\mathcal{M}$ (either empty or of a fixed complex dimension). It makes sense to say that such a submanifold is involutive, or isotropic, or Lagrangian. The *real* dimension of a Lagrangian submanifold Λ of $\mathbb{C}\mathbf{T}^*\mathcal{M}$ is not usually equal to $\mathrm{N} = \dim \mathcal{M}$; what is equal to N is the sum of the (real) rank of the base projection π restricted to Λ and of the complex dimension of $\Lambda \cap \mathbb{C}\mathbf{T}_x \neq \emptyset$. Our basic hypothesis is that the zero-set Σ of the functions $p_1, ..., p_n$ is an involutive submanifold of $\mathbb{C}\mathbf{T}^*\mathcal{M}$ on which the rank of π is equal to N. We also assume that the fibre-differentials $\partial_\xi p_j$ are linearly independent on Σ.

We affix the adjective "fibre-holomorphic" to the various ingredients of this symplectic geometry, e.g., the *fibre-holomorphic Poisson bracket* $\{f, g\}$ of two $\mathscr{C}^1$ functions in $\mathbb{C}\mathbf{T}^*\mathcal{M}$ whose restrictions to the fibres $\mathbb{C}\mathbf{T}^*_x\mathcal{M}$ are holomorphic. Here the Darboux theorem is not generally true. As a matter fact, the main theorem of the paper, Theorem 2.2, states the equivalence of the following two properties: the existence of a set of fibre-holomorphic $\mathscr{C}^\infty$ functions $p_j(x, \xi)$, $q_j(x, \xi)$ $(j = 1, ..., \mathrm{N})$ incorporating the p_j in Eqs. (1) and such that

$$\{p_j, p_k\} = \{q_j, q_k\} = \{p_j, q_k\} - \delta_{jk} = 0, \qquad j, k = 1, ..., \mathrm{N}; \tag{3}$$

the existence of a $\mathscr{C}^\infty$ solution $w(x, \theta)$ of (1) which is holomorphic with respect to θ and whose differential with respect to x is equal to θ at the "central" point x_0 (with θ varying in an open neighborhood, in $\Sigma \cap \mathbb{C}\mathbf{T}^*_{x_0}\mathcal{M}$, of a pre-assigned value θ_0).

Theorem 2.1 equates the solvability of the nonlinear system (1) to the existence of "first integrals" that satisfy the Poisson bracket conditions (3), for the system comprising the Hamiltonian "vector fields" (2) as wel as the Cauchy–Riemann vector fields $\sum a_i \partial/\partial\xi_i$ tangent to Σ. Sections 4 and 5 are devoted to instances in which known results about the integrability of the latter system (such as the Newlander–Nirenberg theorem or recent embedding theorems for CR manifolds) yield strong solvability results for the former, nonlinear, system.

For the sake of simplicity attention is limited to Eqs. (1), which are of the eikonal type. More general equations

$$p_j(x, w, dw) = 0, \qquad j = 1, ..., n, \tag{4}$$

could have been studied as well. It would have required equipping the bundle of complex one-jets with its natural fibre-holomorphic symplectic structure. It is done in Section X.1 of [5]. The extension to one-jets is both routine and cumbersome. At any rate, a well-known trick, based on the introduction of an additional variable x_0, reduces a system (4) to a system (1)—as shown in Section 3, in which the main theorem is interpreted for semilinear equations.

A solvability result for a class of (complex) semilinear PDE in the plane can be found in [3]. The reader interested in the uniqueness aspects of the questions discussed here may be referred to [4] (see also [5, Chap. X]).

1. The Fibre-Holomorphic Category

Throughout the present article $\mathcal{M}$ will denote a $\mathscr{C}^\infty$ manifold; $\dim_{\mathbb{R}} \mathcal{M} = N$; $\mathbb{C}T^*\mathcal{M}$ will denote the complex(ified) cotangent bundle of $\mathcal{M}$ and $\mathbb{C}T\mathcal{M}$ its complex tangent bundle; $\pi\colon \mathbb{C}T^*\mathcal{M} \to \mathcal{M}$ will stand for the base projection. Given an open subset Ω of $\mathbb{C}T^*\mathcal{M}$ we shall call *fibre-holomorphic* any continuous function in Ω whose restriction to $\Omega \cap \mathbb{C}T^*_x\mathcal{M}$ is holomorphic, whatever $x \in \mathcal{M}$. We shall denote by $\mathscr{H}o\ell_f(\Omega)$ the space of fibre-holomorphic functions in Ω.

The differentials of the germs, at arbitrary points of $\mathbb{C}T^*\mathcal{M}$, of the fibre-holomorphic functions of class $\mathscr{C}^1$ (or $\mathscr{C}^\infty$) make up a vector subbundle of $\mathbb{C}T^*(\mathbb{C}T^*\mathcal{M})$ which will be denoted by $\mathscr{T}'^{1,0}(\mathbb{C}T^*\mathcal{M})$. We shall often use local charts in $\mathbb{C}T^*\mathcal{M}$ defined by means of a local chart in $\mathcal{M}$, $(\mathbf{U}, x_1, ..., x_N)$: without exception (and without it being always recalled) $\zeta_1, ..., \zeta_N$ will denote the complex coordinates in the complex cotangent spaces $\mathbb{C}T^*_x\mathcal{M}$ $(x \in \mathbf{U})$ with respect to the basis $dx_1, ..., dx_N$. Over $\mathbb{C}T^*\mathcal{M}|_{\mathbf{U}}$, $\mathscr{T}'^{1,0}(\mathbb{C}T^*\mathcal{M})$ is spanned by $d\zeta_1, ..., d\zeta_N$ together with the pullbacks (via π) of $dx_1, ..., dx_N$.

The orthogonal of $\mathscr{T}'^{1,0}(\mathbb{C}\mathbf{T}^*\mathscr{M})$ for the duality between complex tangent and cotangent vectors at points of $\mathbb{C}\mathbf{T}^*\mathscr{M}$ is the Cauchy–Riemann bundle tangent to the fibres $\mathbb{C}\mathbf{T}^*_x\mathscr{M}$, here denoted by $\mathbf{T}^{0,1}_{\mathrm{f}}(\mathbb{C}\mathbf{T}^*\mathscr{M})$. Over $\mathbb{C}\mathbf{T}^*\mathscr{M}|_{\mathbf{U}}$ (see above), $\mathbf{T}^{0,1}_{\mathrm{f}}(\mathbb{C}\mathbf{T}^*\mathscr{M})$ is spanned by $\partial/\partial\bar{\zeta}_1, ..., \partial/\partial\bar{\zeta}_N$. The dual of $\mathscr{T}'^{1,0}(\mathbb{C}\mathbf{T}^*\mathscr{M})$ is naturally isomorphic to the quotient vector bundle $\mathscr{T}^{1,0}(\mathbb{C}\mathbf{T}^*\mathscr{M}) = \mathbb{C}\mathbf{T}(\mathbb{C}\mathbf{T}^*\mathscr{M})/\mathbf{T}^{0,1}_{\mathrm{f}}(\mathbb{C}\mathbf{T}^*\mathscr{M})$. A section of $\mathscr{T}^{1,0}(\mathbb{C}\mathbf{T}^*\mathscr{M})$ is not a true vector field. But of course, it acts as a true vector field on $\mathscr{C}^1$ functions that are fibre-holomorphic. In the local coordinates x_i, ζ_i over $\mathbf{U}$ it is represented by a vector field

$$\vartheta = \sum_{k=1}^{N} a_k \partial/\partial x_k + b_k \partial/\partial\zeta_k.$$

We shall say that the section represented by ϑ is fibre-holomorphic if the coefficients a_k and b_k are fibre-holomorphic. We use the analogous terminology for sections of $\mathscr{T}'^{1,0}(\mathbb{C}\mathbf{T}^*\mathscr{M})$.

Given any point $\gamma \in \mathbb{C}\mathbf{T}^*\mathscr{M}$ we equip each fibre $\mathscr{T}'^{1,0}_\gamma(\mathbb{C}\mathbf{T}^*\mathscr{M})$ with a complex symplectic structure, as follows. In local coordinates, consider two elements of $\mathscr{T}'^{1,0}_\gamma(\mathbb{C}\mathbf{T}^*\mathscr{M})$, $u_i = \sum_{k=1}^{N} a_{ik}\, dx_k + b_{ik}\, d\zeta_k$ $(i = 1, 2)$. We define

$$\omega'_\gamma(u_1, u_2) = \sum_{k=1}^{N} a_{2k} b_{1k} - b_{2k} b_{1k}. \tag{1.1}$$

It is readily checked that the value of (1.1) does not depend on the choice of the coordinates x_k; and the function $\gamma \to \omega'_\gamma(u_1, u_2)$ is smooth (resp., fibre-holomorphic) if this is true of both sections u_1 and u_2 of $\mathscr{T}'^{1,0}(\mathbb{C}\mathbf{T}^*\mathscr{M})$. The $\mathbb{C}$-bilinear functional ω'_γ on $\mathscr{T}'^{1,0}_\gamma(\mathbb{C}\mathbf{T}^*\mathscr{M}) \times \mathscr{T}'^{1,0}_\gamma(\mathbb{C}\mathbf{T}^*\mathscr{M})$ is nondegenerate and therefore defines an isomorphism $\iota_\gamma : \mathscr{T}^{1,0}_\gamma(\mathbb{C}\mathbf{T}^*\mathscr{M}) \to \mathscr{T}'^{1,0}_\gamma(\mathbb{C}\mathbf{T}^*\mathscr{M})$. The pullback of ω'_γ under the map $\iota_\gamma \times \iota_\gamma$ is a nondegenerate, skew-symmetric $\mathbb{C}$-bilinear functional on $\mathscr{T}^{1,0}_\gamma(\mathbb{C}\mathbf{T}^*\mathscr{M}) \times \mathscr{T}^{1,0}_\gamma(\mathbb{C}\mathbf{T}^*\mathscr{M})$; as γ ranges over $\mathbb{C}\mathbf{T}^*\mathscr{M}$ the functionals ω_γ define a differential two-form on $\mathbb{C}\mathbf{T}^*\mathscr{M}$,

$$\omega = \sum_{k=1}^{N} d\zeta_k \wedge dx_k,$$

to which we refer, in the sequel, as the *fundamental symplectic* (fibre-holomorphic) *two-form*.

If $\mathbf{E}$ is a complex vector bundle over $\mathbb{C}\mathbf{T}^*\mathscr{M}$ and Ω is an open subset of $\mathbb{C}\mathbf{T}^*\mathscr{M}$ we denote by $\mathscr{C}^r(\Omega; \mathbf{E})$ the space of $\mathscr{C}^r$ sections of $\mathbf{E}$ over Ω $(0 \leqslant r \leqslant +\infty)$. Let us then compose the maps

$$\mathscr{H}ol_{\mathrm{f}}(\Omega) \cap \mathscr{C}^\infty(\Omega) \ni u \to du \in \mathscr{C}^\infty(\Omega; \mathscr{T}'^{1,0}(\mathbb{C}\mathbf{T}^*\mathscr{M})) \ni \alpha$$
$$\to \iota^*\alpha \in \mathscr{C}^\infty(\Omega; \mathscr{T}^{1,0}(\mathbb{C}\mathbf{T}^*\mathscr{M}))$$

(ι^* is the pullback map defined by the isomorphisms ι_γ defined by ω_γ). The section of $\mathscr{T}^{1,0}(\mathbb{C}\mathbf{T}^*\mathscr{M})$ corresponding to $u \in \mathscr{H}o\ell_{\mathrm{f}}(\Omega) \cap \mathscr{C}^\infty(\Omega)$ will be called the *fibre-holomorphic Hamiltonian vector field* of u and denoted by $\mathbf{H}_u$. Over the local chart $(\mathbf{U}, x_1, \ldots, x_N)$ its expression is given by

$$\mathbf{H}_u = \sum_{k=1}^{N} (\partial u/\partial \zeta_k)\, \partial/\partial x_k - (\partial u/\partial x_k)\, \partial/\partial \zeta_k.$$

Of course $\mathbf{H}_u$ is a fibre-holomorphic section of $\mathscr{T}^{1,0}(\mathbb{C}\mathbf{T}^*\mathscr{M})$ over Ω. If v is another element of $\mathscr{H}o\ell_{\mathrm{f}}(\Omega) \cap \mathscr{C}^\infty(\Omega)$ the (fibre-holomorphic) *Poisson bracket* of u and v is equal to

$$\{u, v\} = \sum_{k=1}^{N} (\partial u/\partial \zeta_k)\, \partial v/\partial x_k - (\partial u/\partial x_k)\, \partial v/\partial \zeta_k = \mathbf{H}_u v = -\mathbf{H}_v u.$$

All the customary formulas, valid in real symplectic structures, are also valid here: e.g., the Jacobi identity and its consequence

$$[\mathbf{H}_u, \mathbf{H}_v] = \mathbf{H}_{\{u,v\}}. \tag{1.2}$$

The present set-up faithfully mimics the symplectic structure of the real cotangent bundle $\mathbf{T}^*\mathscr{M}$. The only difference is that fibre-holomorphy (of the functions, of the differentials, of the "vector fields") must be preserved at each step.

In this vein we shall deal solely with submanifolds Σ of $\mathbb{C}\mathbf{T}^*\mathscr{M}$ that are fibre-holomorphic: by definition this will mean that, for every $x \in \mathscr{M}$, either $\Sigma \cap \mathbb{C}\mathbf{T}^*_x\mathscr{M}$ is empty or else it is a holomorphic submanifold of $\mathbb{C}\mathbf{T}^*_x\mathscr{M}$ whose complex dimension ν is independent of x. As a consequence the intersection $\mathbf{T}^{0,1}_{\mathrm{f}}(\mathbb{C}\mathbf{T}^*\mathscr{M}) \cap \mathbb{C}\mathbf{T}\Sigma$ is a vector bundle over Σ, of (complex) rank ν. This vector bundle is the tangential Cauchy–Riemann bundle of the "fibres" $\Sigma \cap \mathbb{C}\mathbf{T}^*_x\mathscr{M}$; we call it $\mathbf{T}^{0,1}_{\mathrm{f}}(\Sigma)$. We denote by $\mathscr{T}^{1,0}(\Sigma)$ the image of the natural (injective) bundle map $\mathbb{C}\mathbf{T}\Sigma/\mathbf{T}^{0,1}_{\mathrm{f}}(\Sigma) \to \mathscr{T}^{1,0}(\mathbb{C}\mathbf{T}^*\mathscr{M}) = \mathbb{C}\mathbf{T}(\mathbb{C}\mathbf{T}^*\mathscr{M})/\mathbf{T}^{0,1}_{\mathrm{f}}(\mathbb{C}\mathbf{T}^*\mathscr{M})$. A section of $\mathscr{T}^{1,0}(\Sigma)$ is a section of $\mathscr{T}^{1,0}(\mathbb{C}\mathbf{T}^*\mathscr{M})$ which is tangent to Σ.

In accordance with the terminology in real symplectic geometry we shall say that Σ is *symplectic* if the restriction of ω to each fibre of $\mathscr{T}^{1,0}(\Sigma)$ is nondegenerate; Σ is *isotropic* if the pullback of ω to Σ vanishes identically; Σ is *co-isotropic*, or *involutive*, if, whatever $\gamma \in \Sigma$, the orthogonal $\mathscr{T}^{1,0}_\gamma(\Sigma)^\perp$ of $\mathscr{T}^{1,0}_\gamma(\Sigma)$ in $\mathscr{T}^{1,0}_\gamma(\mathbb{C}\mathbf{T}^*\mathscr{M})$ for the bilinear functional ω_γ is contained in $\mathscr{T}^{1,0}(\Sigma)$. Finally, we say that Σ is *Lagrangian* if Σ is both isotropic and co-isotropic.

EXAMPLE 1.1 Let $h \in \mathscr{C}^\infty(\mathscr{M})$ and let Λ_h denote the range of the section $x \to (x, dh(x))$. The pullback to Λ_h of the one-form $\sigma = \sum_{k=1}^{N} \zeta_k\, dx_k$ is equal to dh; therefore the symplectic two-form $\omega = d\sigma$ vanishes identically on Λ_h: Λ_h is isotropic. Given any $\gamma \in \Lambda_h$ the complex tangent space to Λ_h at γ consists of the lift of the complex tangent vectors to $\mathscr{M}$ at $\pi(\gamma)$. By using local coordinates x_i one sees immediately that it is equal to $\mathscr{T}^{1,0}_\gamma(\Sigma)$ as well as to $\mathscr{T}^{1,0}_\gamma(\Sigma)^\perp$. Thus Λ_h is Lagrangian; dim $\Lambda_h = \text{N}$.

EXAMPLE 1.2. Let $\mathscr{M}'$ be a smooth, closed submanifold of $\mathscr{M}$. We select the local chart $(\mathbf{U}, x_1, ..., x_{\text{N}})$ in such a way that $\mathscr{M}' \cap \mathbf{U} = \{x \in \mathbf{U}; x_{r+1} = \cdots = x_{\text{N}} = 0\}$ $(0 \leqslant r = \dim_{\mathbb{R}} \mathscr{M}' \leqslant \text{N})$. Then the complexified conormal bundle of $\mathscr{M}'$ in $\mathscr{M}$, $\mathbb{C}\mathbf{N}^*\mathscr{M}'$, can be defined over $\mathbf{U}$ by the equations $x_{r+1} = \cdots = x_{\text{N}} = \zeta_1 = \cdots = \zeta_r = 0$. Clearly the pullback of $\omega = \sum_{k=1}^{\text{N}} d\zeta_k \wedge dx_k$ to $\mathbb{C}\mathbf{N}^*\mathscr{M}'|_{\mathbf{U}}$ vanishes; $\mathscr{T}^{1,0}(\mathbb{C}\mathbf{N}^*\mathscr{M}')$ is spanned over $\mathbb{C}\mathbf{N}^*\mathscr{M}'|_{\mathbf{U}}$ by $\partial/\partial x_1, ..., \partial/\partial x_r, \partial/\partial\zeta_{r+1}, ..., \partial/\partial\zeta_{\text{N}}$ and is thus equal to $\mathscr{T}^{1,0}(\mathbb{C}\mathbf{N}^*\mathscr{M}')^\perp$. This proves that $\mathbb{C}\mathbf{N}^*\mathscr{M}'$ is Lagrangian; note that $\dim_{\mathbb{R}} \mathbb{C}\mathbf{N}^*\mathscr{M}' = 2\text{N} - r$.

As Example 1.2 shows the (real) dimension of a Lagrangian submanifold Λ of $\mathbb{C}\mathbf{T}^*\mathscr{M}$ can take any value between N and 2N. Suppose the rank of $\pi|_\Lambda$, the base projection restricted to Λ, is equal to r at every point of Λ and let ν denote the complex dimension of the intersections $\Lambda \cap \mathbb{C}\mathbf{T}^*_x\mathscr{M}$ that are not empty. We always have

$$r + \nu = \text{N}. \tag{1.3}$$

2. Solvability of Systems of First-Order Differential Equations

DEFINITION 2.1. A fibre-holomorphic, $\mathscr{C}^\infty$, closed submanifold Σ of $\mathbb{C}\mathbf{T}^*\mathscr{M}$ will be called a *system of first-order differential equations* (abbreviated below to *systems of DE*) if it satisfies the following two conditions:

$$\text{the base projection } \pi \text{ restricted to } \Sigma \text{ has rank N at every point of } \Sigma; \tag{2.1}$$

$$\Sigma \text{ is co-isotropic.} \tag{2.2}$$

We shall say that the system Σ is holonomic if, moreover, Σ is Lagrangian.

Property (2.1) entails that $\pi(\Sigma)$ is an open subset of $\mathscr{M}$. If furthermore we take into account the fact that Σ is fibre-holomorphic we see that Σ can be covered with open subsets Ω of $\mathbb{C}\mathbf{T}^*\mathscr{M}$ in which there exist functions $p_1, ... p_n \in \mathscr{H}ol_f(\Omega) \cap \mathscr{C}^\infty(\Omega)$ that have the following properties:

$$\Sigma \cap \Omega = \{(x, \zeta) \in \Omega; p_1(x, \zeta) = \cdots = p_n(x, \zeta) = 0\}; \tag{2.3}$$

$$d_\zeta p_1 \wedge \cdots \wedge d_\zeta p_n \neq 0 \textit{ at every point of } \Sigma \cap \Omega. \tag{2.4}$$

The integer $n \leqslant \text{N}$ is the complex codimension in $\mathbb{C}T^*_x\mathcal{M}$ of the holomorphic submanifold $\Sigma \cap \mathbb{C}T^*_x\mathcal{M}$ (for those $x \in \mathcal{M}$ such that the latter is not empty).

The orthogonal of $\mathscr{T}^{1,0}(\Sigma)$ for the symplectic form $\omega = \sum_{k=1}^{\text{N}} d\zeta_k \wedge dx_k$ is spanned, over the above set Ω, by the fibre-holomorphic Hamiltonian "vector fields" $\mathbf{H}_{p_j}$. Condition (2.2) requires that they be tangent to Σ, i.e., $\mathbf{H}_{p_j} p_k \equiv 0$ on $\Sigma \cap \Omega$:

$$p_1 = \cdots = p_n = 0 \Rightarrow \qquad \forall j, k = 1, ..., n, \{p_j, p_k\} = 0. \tag{2.5}$$

Property (2.5) remains valid if we replace $p_1, ..., p_n$ by linear substitutions $p_j^* = \sum_{k=1}^{n} c_{jk} p_k$, with $c_{jk} \in \mathscr{H}ol_{\mathrm{f}}(\Omega) \cap \mathscr{C}^\infty(\Omega)$, $\det(c_{jk})_{1 \leqslant j,k \leqslant n} \neq 0$. Observe that $\dim_{\mathbb{R}} \Sigma = 3\text{N} - 2\text{n}$; when Σ is holonomic, $\dim_{\mathbb{R}} \Sigma = \text{N}$. In the latter case the intersections $\Sigma \cap \mathbb{C}T^*_x\mathcal{M}$ are discrete subsets of $\mathbb{C}T^*\mathcal{M}$, for all $x \in \mathcal{M}$.

A $\mathscr{C}^1$ function w in an open subset U of $\mathcal{M}$ will be called a *solution in* U of the system of DE Σ if $(x, dw(x)) \in \Sigma$ for every $x \in \text{U}$. If the section $\text{U} \ni x \to (x, dw(x))$ is contained in an open subset Ω of $\mathbb{C}T^*\mathcal{M}$ in which the representation (2.3)–(2.4) is valid, we must have

$$p_j(x, dw(x)) = 0, \qquad \forall j = 1, ..., n, \forall x \in \text{U}. \tag{2.6}$$

DEFINITION 2.2. We say that the system of DE Σ is *solvable at the point* $(x_0, \zeta_0) \in \Sigma$ if there is a $\mathscr{C}^\infty$ solution w of Σ in an open neighborhood U of x_0 such that $dw(x_0) = \zeta_0$.

THEOREM 2.1. *For the system of DE Σ to be solvable at $(x_0, \zeta_0) \in \Sigma$ it is necessary and sufficient that Σ contains a holonomic system of DE, Λ, passing through (x_0, ζ_0).*

Proof. Let U and w be as in Definition 2.2. The section $\text{U} \ni x \to (x, dw(x)) \in \Sigma$ is Lagrangian (see Example 1.1) and passes through (x_0, ζ_0). Conversely, let Λ be as in Theorem 2.1: since Λ satisfies Condition (2.1), if the open neighborhood U of x_0 is sufficiently small there is a $\mathscr{C}^\infty$ section $\text{U} \ni x \to (x, \zeta(x)) \in \Lambda$ such that $\zeta(x_0) = \zeta_0$. The pullback to Λ of the symplectic form $\omega = d(\sum_{k=1}^{\text{N}} \zeta_k \, dx_k)$ vanishes identically. By the Poincaré lemma there is a function $w \in \mathscr{C}^\infty(\text{U})$ such that $\zeta = dw$ in U. ∎

COROLLARY 2.1. *A holonomic system of DE is solvable at every one of its points.*

Henceforth Σ denotes a system of DE. We shall deal with fibre-holomorphic functions in an open subset $\mathcal{O}$ of Σ; they are the continuous functions in $\mathcal{O}$ whose restrictions to all intersections $\mathcal{O} \cap \mathbb{C}T^*_x\mathcal{M}$ $(x \in \mathcal{M})$ are holomorphic; they make up a space denoted by $\mathscr{H}ol_{\mathrm{f}}(\mathcal{O})$.

For brevity we call $\mathscr{V}_\Sigma$ the orthogonal $\mathscr{T}^{1,0}(\Sigma)^\perp$ of $\mathscr{T}^{1,0}(\Sigma)$ for the symplectic form ω. As already stated, $\mathscr{V}_\Sigma$ is a vector subbundle of (complex) rank n of $\mathscr{T}^{1,0}(\Sigma)$; when (2.3)–(2.4) is valid, $\mathbf{H}_{p_1}, \ldots, \mathbf{H}_{p_n}$ form a linear basis of $\mathscr{V}_\Sigma$ over $\Sigma \cap \Omega$. From (2.3), (2.4), and (2.5) we derive $\{p_i, p_j\} = \sum_{k=1}^n c_{ijk}\, p_k$, whence, by (1.6),

$$[\mathbf{H}_{p_i}, \mathbf{H}_{p_j}] = \sum_{k=1}^{n} c_{ijk} \mathbf{H}_{p_k} + p_k \mathbf{H}_{c_{ijk}}.$$

Therefore, when restricted to Σ,

$$[\mathbf{H}_{p_i}, \mathbf{H}_{p_j}] = \sum_{k=1}^{n} c_{ijk} \mathbf{H}_{p_k} \qquad (i, j = 1, \ldots, n).$$

In other words, the commutation bracket of two sections of $\mathscr{V}_\Sigma$ (regarded as operators on the fibre-holomorphic $\mathscr{C}^\infty$ functions) is again a section of $\mathscr{V}_\Sigma$. This defines on Σ a *fibre-holomorphic formally integrable structure*, akin to the $\mathscr{C}^\infty$ formally integrable structures as defined in [5].

In the sequel, a $\mathscr{C}^1$ function u in an open subset $\mathcal{O}$ of Σ will be called a *solution of the homogeneous Hamiltonian equations* in $\mathcal{O}$ if $u \in \mathscr{H}ol_{\mathrm{f}}(\mathcal{O})$ and if $\mathbf{L}u = 0$ whatever the section $\mathbf{L}$ of $\mathscr{V}_\Sigma$ over $\mathcal{O}$. Then let γ be an arbitrary point of $\mathcal{O}$ and extend u as a fibre-holomorphic $\mathscr{C}^1$ function $\tilde{u}$ in an open neighborhood Ω of γ in $\mathbb{C}\mathbf{T}^*\mathscr{M}$. Let $p_1, \ldots, p_n$ be as in (2.3)–(2.4). Since u is a solution of the homogeneous Hamiltonian equations, $\mathbf{H}_{p_j}\tilde{u} = -\mathbf{H}_{\tilde{u}}\, p_j$ vanishes identically in $\Sigma \cap \Omega$ $(\subset \mathcal{O})$, whatever $j = 1, \ldots, n$. This means that $\mathbf{H}_{\tilde{u}}$ is a section of $\mathscr{T}^{1,0}(\Sigma)$ over $\Sigma \cap \Omega$, which is easily seen to be independent of the choice of the extension $\tilde{u}$. Patching together these sections yields a section of $\mathscr{T}^{1,0}(\Sigma)$ over $\Sigma \cap \mathcal{O}$, which we denote by $\mathbf{H}_u$. This observation allows us to introduce the fibre-holomorphic Poisson bracket $\{u, g\} = \mathbf{H}_u g$ of the solution u of the homogeneous Hamiltonian equations with an arbitrary function $g \in \mathscr{H}ol_{\mathrm{f}}(\mathcal{O}) \cap \mathscr{C}^1(\mathcal{O})$. When g is also a solution of the homogeneous Hamiltonian equations, $\{u, g\} = -\{g, u\}$.

DEFINITION 2.3. We shall say that the system of DE Σ is *stably solvable at the point* $(x_0, \zeta_0) \in \Sigma$ if the following property holds:

> There exist an open neighborhood $\mathbf{U}$ of x_0 in $\mathscr{M}$, one, $\mathcal{O}$, of ζ_0 in $\Sigma \cap \mathbb{C}\mathbf{T}^*_{x_0}\mathscr{M}$ and a function $w(x, \theta) \in \mathscr{C}^\infty(\mathbf{U} \times \mathcal{O})$, holomorphic with respect to θ, that satisfy the following two conditions: (2.7)

$$\forall x \in \mathbf{U}, \theta \in \mathcal{O}, \qquad (x, d_x w(x, \theta)) \in \Sigma; \tag{2.8}$$

$$\forall \theta \in \mathcal{O}, \qquad d_x w(x_0, \theta) = \theta. \tag{2.9}$$

Below we use the notation $m = \mathrm{N} - n = \dim_{\mathbb{C}}(\Sigma \cap \mathbb{C}\mathrm{T}^*_{x_0}\mathscr{M})$. Note that $\dim_{\mathbb{R}} \Sigma = \mathrm{N} + 2m = 3m + n$.

THEOREM 2.2. *For the system of DE Σ to be stably solvable at (x_0, ζ_0) it is necessary and sufficient that the following equivalent conditions be satisfied:*

There exist an open neighborhood $\widetilde{\mathscr{O}}$ of (x_0, ζ_0) in Σ and $2m$ solutions of the homogeneous Hamiltonian equations in $\widetilde{\mathscr{O}}$, $u_1, ..., u_m, v_1, ..., v_m \in \mathscr{C}^\infty(\widetilde{\mathscr{O}})$, such that (2.10)

$$\{u_j, u_k\} \equiv \{v_j, v_k\} \equiv \{v_j, v_k\} - \delta_{jk} \equiv 0 \qquad (j, k = 1, ..., m). \tag{2.11}$$

There exist an open neighborhood $\widetilde{\mathscr{O}}$ of (x_0, ζ_0) in Σ and m solutions of the homogeneous Hamiltonian equations in $\widetilde{\mathscr{O}}$, $u_1, ..., u_m \in \mathscr{C}^\infty(\widetilde{\mathscr{O}})$, such that (2.12)

$$\{u_j, u_k\} \equiv 0 \qquad (j, k = 1, ..., m); \tag{2.13}$$

the map $\zeta \to u(x, \zeta)$ $(u = (u_1, ..., u_m))$ is a biholomorphism of $\widetilde{\mathscr{O}} \cap \mathbb{C}\mathrm{T}^*_x\mathscr{M}$ onto an open subset of $\mathbb{C}^m$ independent of $x \in \pi(\widetilde{\mathscr{O}})$. (2.14)

Proof. We shall reason above a local chart $(\mathrm{U}, x_1, ..., x_{\mathrm{N}})$, with $x_0 \in \mathrm{U}$. Possibly after relabeling the coordinates x_i we may assume that Σ is defined, in an open neighborhood Ω of (x_0, ζ_0), by equations

$$\zeta_{m+j} = q_j(x, \zeta_1, ..., \zeta_m), \qquad j = 1, ..., n, \tag{2.15}$$

with $q_j(x, \zeta_1, ..., \zeta_m) \in \mathscr{H}ol_\zeta(\Omega) \cap \mathscr{C}^\infty(\Omega)$. By virtue of (2.5), the pullback of ω to $\mathscr{T}^{1,0}(\Sigma)|_\Omega$ is equal to

$$\sum_{i=1}^{m} \left(d\zeta_i - \sum_{k=m+1}^{\mathrm{N}} (\partial q_k/\partial x_i)\, dx_k\right) \wedge \left(dx_i + \sum_{k=m+1}^{\mathrm{N}} (\partial q_k/\partial \zeta_i)\, dx_k\right).$$

A. (2.10) $\Rightarrow$ (2.12). Property (2.10) entails that the pullback of ω to $\mathscr{T}^{1,0}(\Sigma)|_\Omega$ is equal to $\sum_{i=1}^m du_i \wedge dv_i$. Consequently, there is a constant matrix $(\gamma_{i,j})_{i,j=1,...,2m} \in \mathrm{Sp}(m, \mathbb{C})$ such that, at the point (x_0, ζ_0),

$$d\zeta_i - \sum_{k=m+1}^{\mathrm{N}} (\partial q_k/\partial x_i)\, dx_k = \sum_{j=1}^{m} \gamma_{i,j}\, du_j + \gamma_{i,m+j}\, dv_j \qquad (1 \leqslant i \leqslant m).$$

Let us then define the functions in $\widetilde{\mathscr{O}}$,

$$\tilde{u}_i = \sum_{j=1}^{m} \gamma_{i,j} u_j + \gamma_{i,m+j} v_j \qquad (1 \leqslant i \leqslant m).$$

It is clear that the $\tilde{u}_i$ are solutions of the homogeneous Hamiltonian equations and that the Jacobian matrix of $\tilde{u}_1, ..., \tilde{u}_m$ with respect to $\zeta_1, ..., \zeta_m$ is equal, at (x_0, ζ_0), to the $m \times m$ identity matrix I_m. By virtue of (2.11) the Poisson brackets $\{\tilde{u}_j, \tilde{u}_k\}$ are constant in $\tilde{\mathcal{C}}$. But at (x_0, ζ_0), $\{\tilde{u}_j, \tilde{u}_k\} = 0$, hence $\{\tilde{u}_j, \tilde{u}_k\} \equiv 0$ in $\tilde{\mathcal{C}}$. Thus all the conditions in (2.12) are satisfied if we substitue $\tilde{u}_j$ for u_j.

B. (2.7) $\Rightarrow$ (2.12). For fixed $\theta \in \mathcal{C}$, the section $\mathbf{U} \ni x \to (x, d_x w(x, \theta)) \in \Sigma$ describes a Lagrangian submanifold Λ_θ of $\mathbb{C}\mathbf{T}^*\mathcal{M}$ (cf. proof of Theorem 2.1). The Hamiltonian structure bundle $\mathscr{V}_\Sigma$ restricted to Λ_θ is contained in $\mathscr{T}^{1,0}(\Lambda_\theta)$ since $\mathscr{V}_\Sigma|_{\Lambda_\theta} \subset \mathscr{T}^{1,0}(\Lambda_\theta)^-$ and since Λ_θ is co-isotropic. We make use of the fact that $d^2_{x,\theta} w(x_0, \theta) = I_m$ according to (2.9). We may apply the holomorphic implicit function theorem (with $\mathscr{C}^\infty$ dependence on the parameter $x \in \mathcal{M}$) and regard θ as a function of (x, ζ), $u(x, \zeta)$, by solving the equation $\zeta = d_x w(x, \theta)$. It is clear that the neighborhood $\tilde{\mathcal{C}}$ can be chosen to satisfy (2.14). On the other hand, since each function $u_j(x, \zeta)$ is constant (and equal to θ_j, the jth coordinate of θ in ζ-space) on Λ_θ, u_j is annihilated by any section of $\mathscr{V}_\Sigma$. Again due to the fact that Λ_θ is Lagrangian, $\{u_j, u_k\} \equiv 0$ on Λ_θ and therefore $\{u_j, u_k\} \equiv 0$ in a whole neighborhood of (x_0, ζ_0) fibered by submanifolds Λ_θ.

C. (2.12) $\Rightarrow$ (2.7) and (2.10). We avail ourselves of (2.14) to solve the equations $u(x, \zeta) = \theta$ with respect to $\zeta = \zeta(x, \theta)$. As before let Λ_θ denote the submanifold of Σ consisting of the points $(x, \zeta(x, \theta))$. Since the functions u_j are solutions of the homogeneous Hamiltonian equations, the vector bundle $\mathscr{V}_\Sigma$ restricted to Λ_θ is contained in $\mathscr{T}^{1,0}(\Lambda_\theta)$. By virtue of (2.13) each $\mathbf{H}_{u_j}$ $(j = 1, ..., m)$ is also a section of $\mathscr{T}^{1,0}(\Lambda_\theta)$. Let us reason in an open set Ω in which (2.3)–(2.4) holds. The differential $dp_1, ..., dp_n$ span $\mathbb{C}\mathbf{N}^*\Sigma$ over $\Sigma \cap \Omega$; because of this and of (2.14) we must have $du_1 \wedge \cdots \wedge du_m \wedge dp_1 \wedge \cdots \wedge dp_n \neq 0$ at every point of $\Sigma \cap \Omega$ (possibly contracted about one of its points). It follows that $\mathbf{H}_{u_1}, ..., \mathbf{H}_{u_m}, \mathbf{H}_{p_1}, ..., \mathbf{H}_{p_n}$ are linearly independent and span $\mathscr{T}^{1,0}(\Lambda_\theta)$ along Λ_θ (note that the rank of $\mathscr{T}^{1,0}(\Lambda_\theta)$ is equal to $\mathrm{N} = m + n$). But then $\omega \equiv 0$ on Λ_θ since all the fibre-holomorphic Poisson brackets of the functions $u_1, ..., u_m, p_1, ..., p_n$ vanish identically. This means that the pullback of $\sum_{k=1}^{\mathrm{N}} \zeta_k \, dx_k$ to Λ_θ is closed and therefore, by Poincaré's lemma, $\zeta_k = \partial w/\partial x_k$ for some function $w \in \mathscr{C}^\infty(\mathbf{U} \times \mathcal{C})$, holomorphic with respect to the second variable, θ (here $\mathbf{U}$ and $\mathcal{C}$ are neighborhoods as in (2.7)). This proves that (2.7) holds true.

With $w(x, \theta)$ as we have just selected it, set $Z_i = \partial w/\partial \theta_i$, $v_i(x, \zeta) = Z_i(x, u(x, \zeta))$. Let us reason in an open set Ω in which (2.3)–(2.4) are valid. We have

$$\mathbf{H}_{p_j} v_i = \sum_{k=1}^{\mathrm{N}} \left\{ (\partial Z_i/\partial x_k + \sum_{l=1}^{m} (\partial Z_i/\partial \theta_l) \, \partial u_l/\partial x_k) \, \partial p_j/\partial \zeta_k \right.$$
$$\left. - \left(\sum_{l=1}^{m} (\partial Z_i/\partial \theta_l) \, \partial u_l/\partial \zeta_k \right) \partial p_j/\partial x_k \right\}$$

$$= \sum_{k=1}^{N} (\partial p_j/\partial \zeta_k)\, \partial Z_i/\partial x_k + \sum_{l=1}^{m} (\partial Z_i/\partial \theta_l)\{p_j, u_l\}$$

$$= \sum_{k=1}^{N} (\partial p_j/\partial \zeta_k)\, \partial Z_i/\partial x_k = \sum_{k=1}^{N} (\partial p_j/\partial \zeta_k)\, \partial^2 w/\partial x_k \partial \theta_i.$$

If we differentiate Eqs. (2.6) with respect to θ we obtain that the restriction of $\mathbf{H}_{p_j} v_i$ to Λ_θ vanishes identically. As a consequence we can select an open neighborhood $\tilde{\mathscr{O}}$ of (x_0, ζ_0) in Σ, fibered by the Lagrangian submanifolds Λ_θ, in which $\{p_j, v_i\} \equiv 0$ for all $i = 1, ..., m$, $j = 1, ..., n$.

In the coordinate system $x_1, ..., x_N, \theta_1, ..., \theta_m$ in $\tilde{\mathscr{O}}$ we may write

$$\mathbf{H}_{u_i} = \sum_{k=1}^{N} (\partial u_i/\partial \zeta_k)(x, \zeta(x, \theta))\, \partial/\partial x_k, \tag{2.16}$$

whence, in Λ_θ,

$$\{u_i, v_j\} = \mathbf{H}_{u_i} Z_j = \sum_{k=1}^{N} (\partial u_i/\partial \zeta_k)\, \partial^2 w/\partial x_k \partial \theta_j$$

$$= (\partial/\partial \theta_j)\, u_i(x, \zeta(x, \theta)) = \partial \theta_i/\partial \theta_j = \delta_{ij}.$$

As a consequence of this we obtain (always in Λ_θ)

$$\mathbf{H}_{v_j} = \sum_{k=1}^{N} (\partial v_j/\partial \zeta_k)(x, \zeta(x, \theta))\, \partial/\partial x_k + \sum_{l=1}^{m} \{v_j, u_l\}\, \partial/\partial \theta_l$$

$$= \sum_{l=1}^{m} (\partial Z_j/\partial \theta_l) \left\{ \sum_{k=1}^{N} (\partial u_l/\partial \zeta_k)\, \partial/\partial x_k \right\} - \partial/\partial \theta_j$$

$$= \sum_{l=1}^{m} (\partial Z_j/\partial \theta_l)\, \mathbf{H}_{u_l} - \partial/\partial \theta_j$$

by (2.16). We derive

$$\mathbf{H}_{v_j} v_k = \sum_{l=1}^{m} (\partial Z_j/\partial \theta_l)\{u_l, v_k\} - \partial Z_k/\partial \theta_j$$

$$= \partial Z_j/\partial \theta_k - \partial Z_k/\partial \theta_j = 0$$

since $Z_j = \partial w/\partial \theta_j$. This concludes the proof of (2.11) and therefore of (2.10), and of Theorem 2.2. ∎

Remark 2.1. The preceding proof of the entailment (2.12) ⇒ (2.10) shows that the solutions u_i and v_j of the homogeneous Hamiltonian equations, in

Property (2.10), can be selected to satisfy the following requirement, in addition to (2.11):

$$\text{The Jacobian matrix of } u_1, ..., u_m, v_1, ..., v_m \text{ with respect to } \zeta_1, ..., \zeta_m, x_1, ..., x_m \text{ is equal to } I_{2m} \text{ at } (x_0, \zeta_0). \tag{2.17}$$

Equations (2.15) provide us with local representations of Σ that are especially convenient. They suggest that we change the notation for the coordinates: we write t_k instead of x_{m+k} and τ_k instead of ζ_{m+k} ($1 \leqslant k \leqslant n$). Now ζ stands for $(\zeta_1, ..., \zeta_m)$ and the functions p_k in (2.3)–(2.4) have the expressions

$$p_k = \tau_k - q_k(x, t, \zeta) \qquad (1 \leqslant k \leqslant n). \tag{2.18}$$

We have

$$\begin{aligned} \mathbf{H}_{p_k} = \partial/\partial t_k &- \sum_{i=1}^{m} \{(\partial q_k/\partial\zeta_i)\, \partial/\partial x_i - (\partial q_k/\partial x_i)\, \partial/\partial\zeta_i\} \\ &+ \sum_{l=1}^{m} (\partial q_k/\partial t_l)\, \partial/\partial\tau_l, \end{aligned} \tag{2.19}$$

whence

$$\begin{aligned} \{p_k, p_l\} &= \partial q_k/\partial t_l - \partial q_l/\partial t_k \\ &\quad - \sum_{i=1}^{m} \{(\partial q_l/\partial\zeta_i)\, \partial q_k/\partial x_i - (\partial q_l/\partial x_i)\, \partial q_k/\partial\zeta_i\}. \end{aligned}$$

This shows that the Poisson brackets $\{p_i, p_j\}$ are independent of $\tau = (\tau_1, ..., \tau_n)$. Since they vanish when $\tau = q(x, t, \zeta)$ $(q = (q_1, ..., q_n))$, we conclude that, *in Ω*,

$$\{p_k, p_l\} \equiv 0, \qquad \forall k, l = 1, ..., n. \tag{2.20}$$

Combined with (1.2), (2.20), entails that $[\mathbf{H}_{p_k}, \mathbf{H}_{p_l}] \equiv 0$ in Ω.

Let $\mathbf{U}$ be an open subset of $\pi(\Omega)$. That $w \in \mathscr{C}^1(\mathbf{U})$ is a solution of the system Σ means that, in $\mathbf{U}$,

$$\partial w/\partial t_j = q_j(x, t, w_x), \qquad j = 1, ..., n. \tag{2.21}$$

The central point, which lies in $\mathbf{U}$, will now be called (x_0, t_0). We reason about a point $(x_0, t_0, \zeta_0, \tau_0)$ of Σ $(\tau_0 = q(x_0, t_0, \zeta_0))$. We may coordinatize $\Sigma \cap \Omega$ by means of x_i, ζ_i, t_k ($i, j = 1, ..., m$, $k = 1, ..., n$) and identify a point $\theta \in \Sigma \cap \mathbb{C}T^*_{x_0}\mathscr{M}$ to a point in $\mathbb{C}^m$ by means of its ζ-coordinates, $\theta_1, ..., \theta_m$. Condition (2.9) can then be stated as

$$w_x(x_0, \theta) = \theta. \tag{2.22}$$

Next, going to Condition (2.10) we may extend the functions u_i and v_j to Ω (possibly contracted about $(x_0, t_0, \zeta_0, \tau_0)$) as functions independent of τ; such extensions are obviously fibre-holomorphic. By virtue of (2.19) it follows that all the Poisson brackets of u_i, v_j, p_k are independent of τ. But then (2.11) entails, in Ω,

$$\begin{aligned}\{u_i, u_j\} = \{v_i, v_j\} = \{u_i, v_j\} - \delta_{ij} = \{u_i, p_k\}\\ = \{v_j, p_k\} = 0, \qquad \forall i, j = 1, ..., m, k = 1, ..., n.\end{aligned} \tag{2.23}$$

Finally, we observe that, in Ω,

$$\begin{aligned}\{u_i, t_l\} = \{v_j, t_l\} = \{p_k, t_l\} - \delta_{kl}\\ = \{t_k, t_l\} = 0, \qquad \forall i, j = 1, ..., m, k, l = 1, ..., n.\end{aligned} \tag{2.24}$$

In other words, Property (2.10) states that we may find functions $u_i, v_j, p_k, t_l \in \mathscr{H}o\ell_{\mathrm{f}}(\Omega) \cap \mathscr{C}^\infty(\Omega)$ such that, in Ω, the symplectic form ω has the expression

$$\omega = \sum_{i=1}^{m} du_i \wedge dv_i + \sum_{k=1}^{n} dp_j \wedge dt_j. \tag{2.25}$$

This can be seen as a fibre-holomorphic variant of Darboux's theorem.

3. Semilinear Equations

Let $x_1, ..., x_m, t_1, ..., t_n$ denote the coordinates in $\mathbb{R}^{m+n}$. Suppose given n $\mathscr{C}^\infty$ commuting vector fields

$$\mathbf{L}_j = \partial/\partial t_j + \sum_{k=1}^{m} \lambda_{jk}(x, t)\, \partial/\partial x_k \qquad (j = 1, ..., n),$$

as well as n functions $f_1, ..., f_n \in \mathscr{C}^\infty(\mathbb{R}^{m+n} \times \mathbb{C})$ holomorphic with respect to the variable in $\mathbb{C}$. We shall be interested in the system of semilinear equations

$$\mathbf{L}_j u = f_j(x, t, u), \qquad j = 1, ..., n. \tag{3.1}$$

Proposition 3.1. *Assume that the system of linear equations*

$$\mathbf{L}_j \psi + f_j \psi_\eta = 0, \qquad j = 1, ..., n, \tag{3.2}$$

has a $\mathscr{C}^\infty$ solution $\psi(x, t, \eta)$ in an open neighborhood $\mathbf{U}$ of the origin in $\mathbb{R}^{m+n} \times \mathbb{C}$ which is holomorphic with respect to η and is such that $\psi_\eta \neq 0$ at every point of $\mathbf{U}$.

Then the system of semilinear equations (3.1) *has a $\mathscr{C}^\infty$ solution w in an open neighborhood of the origin in $\mathbb{R}^{m+n}$.*

Proof. Our hypothesis allows us to solve the equation $\psi(x, t, \eta) = \psi(0, 0, 0)$ with respect to η; the solution $w(x, t)$ is a $\mathscr{C}^\infty$ function in a neighborhood of the origin in $\mathbb{R}^{m+n}$. Letting $\mathbf{L}_j$ act on both sides of the equation $\psi(x, t, w) = \psi(0, 0, 0)$ yields $(\mathbf{L}_j[\psi(x, t, \eta)] + \psi_\eta(x, t, \eta)\,\mathbf{L}_j w)_{\eta = w} = 0$, that is, by (3.2), $\psi_\eta(x, t, w)(-f_j(x, t, w) + \mathbf{L}_j w) = 0$. Since $\psi_\eta \neq 0$ we obtain (3.1). ∎

If $w(x, t)$ is the solution of (3.1) in the proof of Proposition 3.1 necessarily $w(0, 0) = 0$. Also, since $\psi_x(x, t, w) + \psi_\eta(x, t, w)\, w_x = 0$,

$$w_x(0, 0) = -(\psi_x/\psi_\eta)(0, 0, 0). \tag{3.3}$$

PROPOSITION 3.2. *In addition to the hypothesis in Proposition* 3.1 *assume that the following holds*:

$$\exists Z_1, ..., Z_m \in \mathscr{C}^\infty(\mathbb{R}^{m+n}) \text{ such that the matrix } Z_x = (\partial Z_i/\partial x_j)_{i,j=1,...,m} \text{ is nonsingular and} \tag{3.4}$$

$$\mathbf{L}_j Z_i = 0, \qquad i = 1, ..., m,\ j = 1, ..., n. \tag{3.5}$$

Then the system (3.1) *has a solution w in an open neighborhood of the origin in $\mathbb{R}^{m+n}$ which depends holomorphically on $\theta \in \mathbb{C}^m$ and satisfies*

$$w_x(0, 0, \theta) = \theta. \tag{3.6}$$

Proof. After substituting $Z_x(0, 0)^{-1} Z$ for Z we may assume that $Z_x(0, 0) = I_m$, the $m \times m$ identity matrix. If $\psi(x, t, \eta)$ is the solution of (3.2) in the proof of Proposition 3.1 set

$$\tilde{\psi}(x, t, \eta) = \psi(x, t, \eta) - Z(x, t) \cdot (\psi_\eta(0, 0, 0)\,\theta + \psi_x(0, 0, 0)).$$

Then, at the origin, $\tilde{\psi}_\eta = \psi_\eta$, $\tilde{\psi}_x = -\psi_\eta \theta$, whence $-\tilde{\psi}_x/\tilde{\psi}_\eta = \theta$. In other words, if we use $\tilde{\psi}$ instead of ψ to define the solution w of (3.1), we obtain (3.6). ∎

Next we relate the preceding to the theory in Sections 1 and 2. Let y be an additional real variable and set $\tilde{w}(x, t, y) = yw(x, t)$; (3.1) entails

$$\mathbf{L}_j \tilde{w} = y f_j(x, t, \tilde{w}_y), \qquad j = 1, ..., n. \tag{3.7}$$

Conversely, if $\tilde{w}$ is a $\mathscr{C}^\infty$ solution of (3.7) then the function $w(x, t) = \tilde{w}_y(x, 0, t)$ satisfies (3.1). Write $p_j = \tau_j + \lambda_j(x, t) \cdot \xi - y f_j(x, t, \eta) =$

$\sigma(\mathbf{L}_j)(x, t, \xi) - yf_j(x, t, \eta)$ (η is the "dual" complex variable associated to y); then

$$\begin{aligned}\{p_j, p_k\} &= \{\sigma(\mathbf{L}_j) - yf_j(x, t, \eta), \sigma(\mathbf{L}_k) - yf_k(x, t, \eta)\}\\ &= \sigma([\mathbf{L}_j, \mathbf{L}_k]) + y(\mathbf{L}_k f_j - \mathbf{L}_j f_k + f_j\,\partial f_k/\partial\eta - f_k\,\partial f_j/\partial\eta).\end{aligned}$$

If we are to have $\{p_j, p_k\} = 0$ we must require $[\mathbf{L}_j, \mathbf{L}_k] = 0$, as well as

$$\mathbf{L}_k f_j - \mathbf{L}_j f_k = f_j\,\partial f_k/\partial\eta - f_k\,\partial f_j/\partial\eta, \qquad j, k = 1, ..., n. \tag{3.8}$$

In the coordinates x, y, t, ξ, η on the zero-set Σ of the functions p_j we have

$$\begin{aligned}\mathbf{H}_{p_j} &= \mathbf{L}_j - y(\partial f_j/\partial\eta)\,\partial/\partial y\\ &\quad - \sum_{i=1}^{m}\left(\sum_{k=1}^{m}\xi_k\,\partial\lambda_{jk}/\partial x_i - y\partial f_j/\partial x_i\right)\partial/\partial\xi_i + f_j\partial/\partial\eta.\end{aligned}$$

PROPOSITION 3.3. *Let the functions Z_i $(i = 1, ..., m)$ be as in* (3.4) *and let $\psi(x, t, \eta)$ be a $\mathscr{C}^\infty$ solution of* (3.2) *in an open neighborhood* $\mathbf{U}$ *of the origin in $\mathbb{R}^{m+n}\times\mathbb{C}$ which is holomorphic with respect to η and is such that $\psi_\eta \neq 0$ at every point of* $\mathbf{U}$.

The components $u_1, ... u_m$ of the $\mathbf{m}$-*vector $Z_x^{-1}(\xi + y\psi_x/\psi_\eta)$ and the functions $u_{m+1} = \psi$, $v_1 = Z_1, ..., v_m = Z_m$, $v_{m+1} = y/\psi_\eta$ are solutions of the homogeneous Hamiltonian equations $\mathbf{H}_{p_j}h = 0$ $(1 \leqslant j \leqslant n)$ and satisfy the relations*

$$\{u_i, u_j\} = \{v_i, v_j\} = \{u_i, v_j\} - \delta_{ij} = 0, \qquad i, j = 1, ..., m+1. \tag{3.9}$$

Proof. Since ψ is independent of (y, ξ), (3.2) is equivalent to the system of homogeneous Hamiltonian equations $\mathbf{H}_{p_j}u_{m+1} = 0$, $j = 1, ..., n$. Differentiating (3.2) with respect to η yields

$$(\mathbf{L}_j + f_j\partial/\partial\eta - \partial f_j/\partial\eta)\,\psi_\eta^{-1} = 0, \qquad j = 1, ..., n. \tag{3.10}$$

If then $v_{m+1} = y/\psi_\eta$, (3.10) entails $\mathbf{H}_{p_j}v_{m+1} = 0$; and obviously, $\{u_{m+1}, v_{m+1}\} = 1$. If $v_i = Z_i$ we also have $\{v_i, v_j\} = \{v_i, u_{m+1}\} = \{v_i, v_{m+1}\} = 0$ $(i, j = 1, ..., m)$.

Next we introduce the vector fields $\mathbf{M}_i = \sum_{k=1}^m \mu_{ik}\partial/\partial x_k$ $(i = 1, ..., m)$, where μ_{ik} is the generic entry of the matrix Z_x^{-1}. Since the vector fields $\mathbf{M}_i$ and $\mathbf{L}_j$ commute we derive from (3.2):

$$\psi_x^{-1}(\mathbf{L}_j + f_j\partial/\partial\eta)(\mathbf{M}_i\psi) + \mathbf{M}_i f_j = 0, \qquad i = 1, ..., m,\ j = 1, ..., n. \tag{3.11}$$

Szt $u_i = \sigma(\mathbf{M}_i) + y\mathbf{M}_i\psi/\psi_\eta$; then

$$\begin{aligned}\mathbf{H}_{p_j} u_i &= y(\mathbf{M}_i f_j + \mathbf{L}_j(\psi_\eta^{-1}\mathbf{M}_i\psi) - \psi_\eta^{-1}(\partial f_j/\partial\eta)\,\mathbf{M}_i\psi + f_j(\psi_\eta^{-1}\mathbf{M}_i\psi)_\eta) \\ &= y(\mathbf{M}_i\psi)(\mathbf{L}_j(\psi_\eta^{-1}) + f_j(\psi_\eta^{-1})_\eta - \psi_\eta^{-1}\,\partial f_j/\partial\eta) \\ &\quad + (\mathbf{M}_i f_j + \psi_\eta^{-1}(\mathbf{L}_j + f_j\partial/\partial\eta)(\mathbf{M}_i\psi)),\end{aligned}$$

which vanishes identically by virtue of (3.10) and (3.11).

Suppose $1 \leqslant i, j \leqslant m$. Then $\{u_i, v_j\} = \{\sigma(\mathbf{M}_i), Z_j\} = \delta_{ij}$. Since the vector fields $\mathbf{M}_i$ commute pairwise, we obtain

$$\begin{aligned}\{u_i, u_j\} &= \{\sigma(\mathbf{M}_i), y\mathbf{M}_j\psi/\psi_\eta\} - \{\sigma(\mathbf{M}_j), y\mathbf{M}_i\psi/\psi_\eta\} \\ &\quad + \{y\mathbf{M}_i\psi/\psi_\eta, y\mathbf{M}_j\psi/\psi_\eta\} \\ &= y(\mathbf{M}_i(\mathbf{M}_j\psi/\psi_\eta) - \mathbf{M}_j(\mathbf{M}_i\psi/\psi_\eta) \\ &\quad + (\mathbf{M}_i\psi/\psi_\eta)_\eta\,\mathbf{M}_j\psi/\psi_\eta - \mathbf{M}_i\psi/\psi_\eta(\mathbf{M}_j\psi/\psi_\eta)_\eta) \\ &= y((\mathbf{M}_j\psi)\,\mathbf{M}_i(\psi_\eta^{-1}) - (\mathbf{M}_i\psi)\,\mathbf{M}_j(\psi_\eta^{-1}) \\ &\quad + (\mathbf{M}_i\psi_\eta)(\mathbf{M}_j\psi)/\psi_\eta^2 - (\mathbf{M}_i\psi)(\mathbf{M}_j\psi_\eta)/\psi_\eta^2) = 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}\{u_i, v_{m+1}\} &= -\mathbf{H}_{v_{m+1}} u_i = \psi_\eta^{-1}\,\partial u_i/\partial\eta + y(\mathbf{M}(\psi_\eta^{-1}) - (\psi_\eta^{-1})_\eta\,\partial u_i/\partial y) \\ &= y(\psi_\eta^{-1}((\psi_\eta^{-1}\mathbf{M}_i\psi)_\eta - \psi_\eta^{-1}(\mathbf{M}_i\psi)(\psi_\eta^{-1})_\eta + \psi_\eta\mathbf{M}_i(\psi_\eta^{-1})) \\ &= y(\psi_\eta^{-2}(\mathbf{M}_i\psi)_\eta + \mathbf{M}_i(\psi_\eta^{-1})) = 0.\end{aligned}$$

Finally, we look at

$$\{u_i, u_{m+1}\} = -\mathbf{H}_{u_{m+1}} u_i = (\psi_x\partial/\partial\xi - \psi_\eta\partial/\partial y)(\sigma(\mathbf{M}_i) + y\mathbf{M}_i\psi/\psi_\eta) = 0. \quad \blacksquare$$

The stable solvability of the semilinear equation (3.1) entails the local integrability of the system $(\mathbf{L}_1, ..., \mathbf{L}_n)$ (i.e., the existence of the "first integrals" $Z_1, ..., Z_m$).

If the semilinear system (3.1) is linear, i.e., if $f_j(x, t, \eta) = c_j(x, t)\,\eta + g_j(x, t)$, $j = 1, ..., n$, the "compatibility conditions" (3.8) split into two sets of conditions:

$$\mathbf{L}_k c_j = \mathbf{L}_j c_k, \qquad \mathbf{L}_k g_j - \mathbf{L}_j g_k = c_k g_j - c_j g_k, \qquad j, k = 1, ..., n. \tag{3.12}$$

The stable solvability of the linear system (3.1) is equivalent to the conjunction of the local integrability of the system of vector fields $(\mathbf{L}_1, ..., \mathbf{L}_n)$ and of the local solvability of the equations $\mathbf{L}_j u - c_j u = g_j$, $j = 1, ..., n$.

4. Applications to Nonlinear Equations

4.1. *The Real-Analytic Case*

When $\mathscr{M}$ and Σ are real-analytic, Theorem 2.2 yields the classical result that the system of DE Σ is stably solvable at each of its points. It suffices to select the functions p_j in (2.3)-(2.4) to be real-analytic (but not necessarily real-valued!) and extend them as holomorphic functions in some neighborhood of (x_0, ζ_0). Then the holomorphic version of Darboux's theorem yields functions u_i and v_i $(i = 1, ..., m)$ as required in (2.10).

4.2. *The Hypocomplex Case*

We recall the following definition [5, Sect. III.5]. Let $\mathscr{X}$ be a $\mathscr{C}^\infty$ manifold, $\dim \mathscr{X} = p + q$. A smooth vector subbundle $\mathbf{T}'$ of $\mathbb{C}\mathbf{T}^*\mathscr{X}$ of rank p is said to define a *hypocomplex* structure on $\mathscr{X}$ if $\mathscr{X}$ can be covered with open sets $\mathbf{U}$ in which there are p $\mathscr{C}^\infty$ functions $Z_1, ..., Z_p$ whose differentials span $\mathbf{T}'$ over $\mathbf{U}$ and have the following property: *if the differential dh of a* $\mathscr{C}^1$ *function h in an open neighborhood* $\mathbf{U}' \subset \mathbf{U}$ *of some point* $x_0 \in \mathbf{U}$ *is a section of* $\mathbf{T}'$ *over* $\mathbf{U}'$, *then there is a holomorphic function* $\tilde{h}$ *in an open neighborhood of* $Z(x_0)$ *in* $\mathbb{C}^p$ $(Z = (Z_1, ..., Z_p))$ *such that* $h = \tilde{h} \circ Z$ *in a neighborhood of* x_0 *in* $\mathbf{U}'$.

In accordance with classical terminology one may say that a vector subbundle $\mathbf{T}'$ of $\mathbb{C}\mathbf{T}^*\mathscr{X}$ defines an *elliptic structure* on $\mathscr{X}$ if $\mathbf{T}'$ is formally integrable (i.e., if the differential of any smooth section of $\mathbf{T}'$ is a section of the ideal in the exterior algebra $\Lambda\mathbb{C}\mathbf{T}^*\mathscr{X}$ generated by $\mathbf{T}'$) and if, moreover, $\mathbf{T}' \cap \bar{\mathbf{T}}' = 0$. If we introduce the orthogonal $\mathscr{V} \subset \mathbb{C}\mathbf{T}\mathscr{X}$ of $\mathbf{T}'$ for the duality between complex tangent and cotangent vectors, ellipticity of the structure defined by $\mathbf{T}'$ (or by $\mathscr{V}$) is equivalent to the fact that $\mathscr{V}$ satisfies the Frobenius condition (i.e., the commutation bracket of two smooth sections of $\mathscr{V}$ is a section of $\mathscr{V}$) and that the characteristic set of $\mathscr{V}$ (i.e., the intersection of $\mathbf{T}'$ with the real cotangent bundle $\mathbf{T}^*\mathscr{X}$) is the zero-section. It is a consequence of the Newlander–Nirenberg theorem and of Weyl's lemma that *every elliptic structure is hypocomplex* (see [5, Chap. VI]; in Chap. III of the same book the reader will find nonelliptic examples of hypocomplex structures).

We return to the system of DE Σ. Recall that $\mathscr{V}_\Sigma$ is a vector subbundle of $\mathbb{C}\mathbf{T}\Sigma/\mathbf{T}^{0,1}_{\mathrm{f}}(\Sigma)$. Denote by $\mathscr{V}_\Sigma + \mathbf{T}^{0,1}_{\mathrm{f}}(\Sigma)$ the preimage of $\mathscr{V}_\Sigma$ under the quotient map $\mathbb{C}\mathbf{T}\Sigma \to \mathbb{C}\mathbf{T}\Sigma/\mathbf{T}^{0,1}_{\mathrm{f}}(\Sigma)$. Let Ω be an open subset of $\mathbb{C}\mathbf{T}^*\mathscr{M}$ in which (2.3)–(2.4) is valid. Assume furthermore that there are coordinates x_i, t_j in $\pi(\Omega)$ such that the functions p_k are given by (2.18). The vector fields

$$\vartheta_i = \partial/\partial\zeta_i - \sum_{k=1}^{n} (\partial q_k/\partial\zeta_i)(x, t, \zeta)\, \partial/\partial\tau_k \qquad (i = 1, ..., m)$$

are tangent to the intersections $\Sigma \cap \mathbb{C}\mathbf{T}^*_{(x,t)}\mathcal{M}$, $(x, t) \in \pi(\Omega)$. A linear basis of $\mathcal{V}_\Sigma + \mathbf{T}^{0,1}_{\mathrm{f}}(\Sigma)$ over $\Sigma \cap \Omega$ consists of the vector fields

$$\mathbf{H}_{p_1}, ..., \mathbf{H}_{p_n}, \mathcal{G}_1, ..., \mathcal{G}_m. \tag{4.1}$$

Recall that $\dim_{\mathbb{R}} \Sigma = n + 3m$; the local integrability of $\mathcal{V}_\Sigma + \mathbf{T}^{0,1}_{\mathrm{f}}(\Sigma)$ requires that there be $2m$ $\mathscr{C}^\infty$ functions $f_1, ..., f_{2m} \in \mathscr{H}ol_{\mathrm{f}}(\Sigma \cap \Omega) \cap \mathscr{C}^\infty(\Sigma \cap \Omega)$ such that, at every point of $\Sigma \cap \Omega$,

$$\mathbf{H}_{p_k} f_j = 0, \qquad j = 1, ..., 2m, k = 1, ..., n; \tag{4.2}$$

$$df_1 \wedge \cdots \wedge df_{2m} \neq 0. \tag{4.3}$$

THEOREM 4.1. *Suppose that $\mathcal{V}_\Sigma + \mathbf{T}^{0,1}_{\mathrm{f}}(\Sigma)$ defines on Σ a hypocomplex structure. Then Σ is stably integrable at every one of its points.*

Proof. Let $f_1, ..., f_{2m} \in \mathscr{H}ol_{\mathrm{f}}(\Sigma \cap \Omega) \cap \mathscr{C}^\infty(\Sigma \cap \Omega)$ satisfy (4.2) and (4.3). By virtue of the Jacobi identity we have

$$\mathbf{H}_{p_k}\{f_i, f_j\} \equiv 0, \qquad k = 1, ..., n.$$

By the hypocomplexity hypothesis (and provided Ω is sufficiently small) there is a holomorphic function A_{ij} in an open neighborhood $\tilde{\mathcal{O}}$ of $f(\Sigma \cap \Omega)$ in $\mathbb{C}^{2m}$ $(f = (f_1, ..., f_{2m}))$ such that, in $\Sigma \cap \Omega$, $\{f_i, f_j\} = A_{ij}(f)$. By Condition (4.3), $\det(A_{ij}) \neq 0$. Call (B_{ij}) the inverse of the matrix (A_{ij}); the two-form

$$\tilde{\omega} = \sum_{i,j=1}^{2m} B_{ij}(z)\, dz_i \wedge dz_j$$

defines a complex symplectic structure on $\tilde{\mathcal{O}}$. The holomorphic Darboux theorem entails the existence of complex symplectic coordinates, i.e., holomorphic functions $\tilde{u}_i, \tilde{v}_j$ $(1 \leqslant i, j \leqslant m)$ in $\tilde{\mathcal{O}}$ (possibly after further contractions of Ω and $\tilde{\mathcal{O}}$) such that

$$\sum_{i,j=1}^{2m} A_{ij}(z)((\partial \tilde{u}_k/\partial z_i)\, \partial \tilde{v}_l/\partial z_j - (\partial \tilde{u}_k/\partial z_j)\, \partial \tilde{v}_l/\partial z_i) = \delta_{kl} \qquad (1 \leqslant k, l \leqslant m).$$

If we take $u_i = \tilde{u}_i(f)$, $v_j = \tilde{v}_j(f)$ all requirements in (2.10) are satisfied. ∎

It is traditional terminology to say that the system of DE Σ is *elliptic at a point* $\gamma \in \Sigma$ if the linearized system at γ is elliptic, i.e., if the pushdown of $(\mathcal{V}_\Sigma)_\gamma$ under the base projection is elliptic, in the sense used earlier, i.e., in the sense that

$$\pi_*((\mathcal{V}_\Sigma)_\gamma) + \overline{\pi_*((\mathcal{V}_\Sigma)_\gamma)} = \mathbb{C}\mathbf{T}_{\pi(\gamma)}\mathcal{M}. \tag{4.4}$$

Ellipticity is a stable property: if valid at γ it is valid at all nearby points. Suppose (2.3)–(2.4) is valid, with $\gamma \in \Omega$, and let $x_1, ..., x_m, t_1, ..., t_n$ be local coordinates in $\pi(\Omega)$ such that functions p_k have the expressions (2.18); then $\gamma = (x_0, t_0, \zeta_0, \tau_0)$. Property (4.4) is equivalent to the fact that the constant coefficients operators

$$\partial/\partial t_k - \sum_{i=1}^{n} (\partial q_k/\partial \zeta_i)(x_0, t_0, \zeta_0)\, \partial/\partial x_i \qquad (k = 1, ..., n) \tag{4.5}$$

form an elliptic system [(4.5) are the linearizations of the p_j at $(x_0, t_0, \zeta_0, \tau_0)$]. Comparing with (2.20) shows that the ellipticity of the system (4.5) is equivalent to that of the system (4.1) in a neighborhood of $(x_0, t_0, \zeta_0, \tau_0)$ in Σ. Thus, *for Σ to be elliptic at γ it is necessary and sufficient that $\mathscr{V}_\Sigma + \mathbf{T}_{\mathrm{f}}^{0,1}(\Sigma)$ be elliptic in some neighborhood of γ in Σ.* The latter entails that $\mathscr{V}_\Sigma + \mathbf{T}_{\mathrm{f}}^{0,1}(\Sigma)$ is hypocomplex in a neighborhood of γ, whence

COROLLARY 4.1. *If Σ is elliptic at a point γ it is stably solvable at γ.*

When the system under study is holonomic, i.e., when the submanifold Σ is Lagrangian, $\mathbf{T}_{\mathrm{f}}^{0,1}(\Sigma) = 0$ and $\mathscr{V}_\Sigma = \mathbb{C}\mathbf{T}\Sigma$: the system is clearly elliptic, and, in this case, Corollary 4.1 is essentially a restatement of Corollary 2.1.

In the next section we investigate a class of nonelliptic systems to which Theorem 4.1 applies.

4.3. *The Case of a Single Space Variable*

This is the case $m = 1$. If (2.18) is valid we have, in the coordinates x, $t_1, ..., t_n, \zeta$ on Σ,

$$\mathbf{H}_{p_j} = \partial/\partial t_j - (\partial q_j/\partial \zeta)(x, t, \zeta)\, \partial/\partial x + (\partial q_j/\partial x)(x, t, \zeta)\, \partial/\partial \zeta,$$

Condition (2.13) is void, and (2.12) means that the involutive structure defined on Σ by the system of vector fields $\mathbf{H}_{p_1}, ..., \mathbf{H}_{p_n}$, $\partial/\partial \bar{\zeta}$ is locally integrable.

5. A SOLVABILITY CRITERION BASED ON THE LEVI FORM

By the *characteristic set of the system of DE Σ* we shall mean the characteristic set of the formally integrable structure on Σ, $\mathscr{V}_\Sigma + \mathbf{T}_{\mathrm{f}}^{0,1}(\Sigma)$; we shall denote it by $\mathscr{C}\mathit{har}\,\Sigma$. By the *Levi form of the system Σ* at a point $\hat{\gamma} \in \mathscr{C}\mathit{har}\,\Sigma$ we shall mean the Levi form of $\mathscr{V}_\Sigma + \mathbf{T}_{\mathrm{f}}^{0,1}(\Sigma)$ at the point $\hat{\gamma}$.

THEOREM 5.1. *Suppose that the structure on Σ defined by $\mathscr{V}_\Sigma + \mathbf{H}_{\mathrm{f}}^{0,1}(\Sigma)$ is locally integrable and that, at any point $\hat{\gamma} \in \mathscr{C}\mathit{har}\,\Sigma$, the Levi matrix of Σ*

has at least one eigenvalue <0. *Then* Σ *is stably solvable at every one of its points.*

Proof. It is a consequence of Theorem 6.1, Chap. II, in [1] that if the Levi matrix of the structure on Σ defined by $\mathscr{V}_\Sigma + \mathbf{T}^{0,1}_{\mathrm{f}}(\Sigma)$ has at least one eigenvalue <0 at every characteristic point of $\mathscr{V}_\Sigma + \mathbf{T}^{0,1}_{\mathrm{f}}(\Sigma)$ then the structure is hypocomplex. It suffices then to apply Theorem 4.1. ∎

If the Levi matrix of the structure on Σ defined by $\mathscr{V}_\Sigma + \mathbf{T}^{0,1}_{\mathrm{f}}(\Sigma)$ has at least one eigenvalue <0 at every characteristic point of $\mathscr{V}_\Sigma + \mathbf{T}^{0,1}_{\mathrm{f}}(\Sigma)$, then, by antipodality, it also has one eigenvalue >0 at every such point.

Let $\Omega, p_1, ..., p_n$ be as in (2.3), (2.4), and (2.18). We shall use the notation

$$\mathbf{L}_k = \partial/\partial t_k - \sum_{i=1}^{m} (\partial q_k/\partial \zeta_i)(x, t, \zeta)\, \partial/\partial x_i,$$

$$\mathbf{V}_k = \sum_{i=1}^{m} (\partial q_k/\partial x_i)(x, t, \zeta)\, \partial/\partial \zeta_i.$$

We have, on $\Sigma \cap \Omega$ (in the coordinates $x_i, \zeta_j\, t_k$),

$$\mathbf{H}_{p_k} = \mathbf{L}_k + \mathbf{V}_k \qquad (k = 1, ..., n). \tag{5.1}$$

Note also that, in those coordinates,

$$\vartheta_i = \partial/\partial \zeta_i \qquad (i = 1, ..., m). \tag{5.2}$$

If (for fixed ζ) we regard the $\mathbf{L}_k$ as vector fields in the base, we see that they must commute, since the $\mathbf{H}_{p_k}$ do.

We need to deal with functions defined in subsets of the real cotangent bundle of Σ, $\mathbf{T}^*\Sigma$. If $\gamma \in \Sigma \cap \Omega$ we use the coordinates $\hat{x}_i, \hat{\zeta}_j, \hat{t}_k$ in $\mathbf{T}^*_\gamma\Sigma$ with respect to the basis $\{dx_i, d\zeta_j, dt_k\}_{1 \leqslant i,j \leqslant m, 1 \leqslant k \leqslant n}$. Thanks to (5.1) and (5.2) we see that the characteristic set of the system (4.1) is defined by the equations

$$\hat{\zeta}_i = \sigma(\mathbf{L}_k) = 0, \qquad i = 1, ..., m, k = 1, ..., n. \tag{5.3}$$

Here $\sigma(\mathbf{L}_k)$ is the symbol of $\mathbf{L}_k$:

$$\sigma(\mathbf{L}_k) = \hat{t}_k - \sum_{i=1}^{m} \hat{x}_i (\partial q_k/\partial \zeta_i)(x, t, \zeta).$$

The complex equations (5.3) split into $2(m+n)$ real equations, possibly independent.

Any submanifold $\zeta = \text{const}$ of $\Sigma \cap \Omega$, Λ^ζ, is diffeomorphic to an open neighborhood $\mathbf{U}^\zeta$ of (x_0, t_0) in $\mathscr{M}$ via the base projection π. The pushdown of $(\mathscr{V}_\Sigma + \mathbf{T}_f^{0,1}(\Sigma))|_{\Lambda^\zeta}$ via π is the vector subbundle $\mathscr{V}_\Sigma^\zeta$ of $\mathbb{C}\mathbf{T}\mathscr{M}|_{\mathbf{U}^\zeta}$ spanned by the vector fields $\mathbf{L}_k$ (which commute); $\mathscr{V}_\Sigma^\zeta$ defines a formally integrable structure on $\mathbf{U}^\zeta$. The characteristic set of $\mathscr{V}_\Sigma + \mathbf{T}_f^{0,1}(\Sigma)$ at a point $\gamma \in \Lambda^\zeta$ and that of $\mathscr{V}_\Sigma^\zeta$ at $\pi(\gamma)$ can be identified.

To evaluate the Levi form at $\hat{\gamma} \in \mathscr{C}\!\mathit{har}\,\Sigma$ consider the Poisson brackets, evaluated at $\hat{\gamma}$,

$$\{\hat{\zeta}_i, \bar{\hat{\zeta}}_j\},\ \{\hat{\zeta}_j, \overline{\sigma(\mathbf{H}_{p_k})}\},\ \{\sigma(\mathbf{H}_{p_k}), \overline{\sigma(\mathbf{H}_{p_l})}\} \qquad (i, j = 1, ..., m,\ k, l = 1, ..., n).$$

Of course $\{\hat{\zeta}_i, \bar{\hat{\zeta}}_j\} \equiv 0$. On the other hand, $\sigma(\mathbf{H}_{p_k}) = \sigma(\mathbf{L}_k) + \sigma(\mathbf{V}_k)$. Due to the fact that $\mathbf{V}_k$ is a vector field of type $(1, 0)$, with holomorphic coefficients, tangent to the intersections $\Sigma \cap \mathbb{C}\mathbf{T}^*_{(x,t)}\mathscr{M}$, we also have, *in the set* (5.3),

$$\{\hat{\zeta}_i, \overline{\sigma(\mathbf{V}_k)}\} \equiv \{\sigma(\mathbf{L}_k), \overline{\sigma(\mathbf{V}_l)}\} \equiv \{\sigma(\mathbf{V}_k), \overline{\sigma(\mathbf{V}_l)}\} \equiv 0.$$

Denote by F the $m \times n$ matrix with entries $(2\iota)^{-1}\{\bar{\hat{\zeta}}_i, \sigma(\mathbf{L}_k)\}$ $(\iota = \sqrt{-1})$ and by G the $n \times n$ matrix with entries $(2\iota)^{-1}\{\sigma(\mathbf{L}_k), \overline{\sigma(\mathbf{L}_l)}\}$. We may represent the Levi form at $\hat{\gamma} \in \mathscr{C}\!\mathit{har}\,\Sigma$ by the $\mathrm{N} \times \mathrm{N}$ self-adjoint matrix

$$\mathscr{L} = \begin{bmatrix} 0 & F^* \\ F & G \end{bmatrix}.$$

Note that $\frac{1}{2}\{\bar{\hat{\zeta}}_i, \sigma(\mathbf{L}_k)\} = (\partial/\partial\zeta_i)\,\sigma(\mathbf{L}_k)$. If we represent a vector in $\mathbb{C}^{\mathrm{N}}$ as a "column" $\left[\begin{smallmatrix}\mathbf{u}\\ \mathbf{v}\end{smallmatrix}\right]$ with $\mathbf{u} \in \mathbb{C}^m$ and $\mathbf{v} \in \mathbb{C}^n$, we obtain the *Levi quadratic form*

$$\frac{1}{2}\mathscr{L}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix} \cdot \begin{bmatrix}\bar{\mathbf{u}}\\ \bar{\mathbf{v}}\end{bmatrix} = \mathscr{R}e(F\mathbf{u} \cdot \bar{\mathbf{v}}) + \frac{1}{2}G\mathbf{v} \cdot \bar{\mathbf{v}}. \tag{5.4}$$

PROPOSITION 5.1. *For the Levi form of Σ to be positive-semidefinite at a point $\hat{\gamma} \in \mathscr{C}\!\mathit{har}\,\Sigma$ it is necessary and sufficient that $F = 0$ and that G be positive-semidefinite.*

Proof. The condition is obviously sufficient. Conversely, suppose $F \neq 0$ and select $\mathbf{v}$ such that $F^*\mathbf{v} \neq 0$, $\mathbf{u} = -\rho F^*\mathbf{v}$. Then the value of the right-hand side, which is equal to $\frac{1}{2}G\mathbf{v} \cdot \mathbf{v} - \rho\,\|F^*\mathbf{v}\|^2$, will be < 0 as soon as $\rho > 0$ is sufficiently large. ∎

A restatement of Proposition 5.1 is that, for the Levi form of Σ to have at least one eigenvalue < 0 at a characteristic point $\hat{\gamma} \in \mathscr{C}\!\mathit{har}\,\Sigma$, it is necessary and sufficient, at the point $\hat{\gamma}$, either that $F \neq 0$ or, if $F = 0$, that the Levi matrix G of the system $\{\mathbf{L}_1, ..., \mathbf{L}_n\}$ have at least one eigenvalue < 0.

We recall that a vector subbundle $\mathscr{V}$ of $\mathbb{C}\mathbf{T}\mathscr{X}$ is said to define a *CR structure* on the $\mathscr{C}^\infty$ manifold $\mathscr{X}$ if it is formally integrable and if

$\mathcal{V} \cap \bar{\mathcal{V}} = 0$. The structure is said to be of *hypersurface type* if the rank of $\mathcal{V} + \bar{\mathcal{V}}$ is equal to $\dim_{\mathbb{R}} \mathcal{X} - 1$.

PROPOSITION 5.2. *The following two conditions are equivalent*:

There is an open neighborhood $\mathbf{U}$ *of* (x_0, t_0) *in which the vector fields* $\mathbf{L}_j|_{\zeta=\zeta_0}$ $(j = 1, ..., n)$ *define a CR structure.* (5.5)

There is an open neighborhood $\mathcal{O}$ *of* (x_0, t_0, ζ_0) *in* $\Sigma \cap \Omega$ *in which the vector fields* (4.1) *define a CR structure.* (5.6)

Proof. The expressions (5.1) and (5.2) show that the fact that the vector fields (4.1) and their complex conjugates are $\mathbb{C}$-linearly independent is equivalent to the fact that the vector fields $\mathbf{L}_1, ..., \mathbf{L}_n$ are $\mathbb{C}$-linearly independent [if the latter property holds at (x_0, t_0, ζ_0) it also holds at nearby points (x, t, ζ)]. ∎

The example below shows how results about the local integrability (and hypocomplexity) of the system (5.1)–(5.2), when it defines a CR structure of hypersurface type, can be applied to prove the strong solvability of the associated system of nonlinear DE.

EXAMPLE 5.1. We shall look at the following system of DE Σ in $\mathbb{R}^{2n+1}$. We denote by x_i, y_i $(i, j = 1, ..., n)$ and s the coordinates in $\mathbb{R}^{2n+1}$ and by ξ_i, η_j, and σ the dual *complex* coordinates; we write $z_i = x_i + \iota y_i$ $(\iota = \sqrt{-1})$. We take Σ to be the zero-set of the following n functions in $\mathbb{R}^{2n+1} \times \mathbb{C}^{2n+1}$,

$$p_k(z, s, \xi, \eta, \sigma) = \tfrac{1}{2}(\xi_k + \iota\eta_k - a_k\sigma^2) + \iota b_k(z)\,\sigma + f_k(z, s),$$

where $a_k \in \mathbb{C}$, $b_k(z)$ is a $\mathscr{C}^\infty$ function in $\mathbb{C}^n$, $b_k(0) = 0$, and $f_k(z, s)$ is a $\mathscr{C}^\infty$ function in $\mathbb{C}^n \times \mathbb{R}$ $(k = 1, ..., n)$. We shall assume that the vector $\mathbf{a} = (a_1, ..., a_n)$ is $\neq 0$; it will be more precisely chosen below. If we write $\mathbf{L}_k^0 = \partial/\partial\bar{z}_k - \iota b_k(z)\,\partial/\partial s$, the differential equations will be

$$\mathbf{L}_k^0 w = \tfrac{1}{2}a_k w_s^2 + f_k(z, s), \qquad k = 1, ..., n. \tag{5.7}$$

Let us use the coordinates x_i, y_j, $\xi_k - \iota\eta_k$ $(i, j, k = 1, ..., n)$ and σ on Σ; then (5.1) will be valid if $\mathbf{L}_k = \mathbf{L}_k^0 - a_k\sigma\partial/\partial s$ and

$$\mathbf{V}_k = \sum_{j=1}^{n} (\iota\sigma\,\partial b_k/\partial z_j + \partial f_k/\partial z_j)(\partial/\partial\xi_j + \iota\partial/\partial\eta_j) + (\partial f_k/\partial s)\,\partial/\partial\sigma.$$

We shall require, for all $i, j = 1, ..., n$,

$$\partial b_i/\partial\bar{z}_j = \partial b_j/\partial\bar{z}_i, \tag{5.8}$$

$$\mathbf{L}_i f_j = \mathbf{L}_j f_i. \tag{5.9}$$

Condition (5.8) is equivalent to the fact that $[\mathbf{L}_i, \mathbf{L}_j]=0$, also to the fact that there exists a $\mathscr{C}^\infty$ function $\varphi \in \mathscr{C}^\infty(\mathbb{C}^2)$ such that $b_i=\partial\varphi/\partial\bar{z}_i$; (5.8) and (5.9) together ensure that $\{p_i, p_j\}\equiv 0$. Actually (5.9) subdivides into two conditions:

$$\mathbf{L}_j^0 f_i=\mathbf{L}_i^0 f_j, \tag{5.10}$$

$$a_i\,\partial f_j/\partial s=a_j\,\partial f_i/\partial s. \tag{5.11}$$

Since $\mathbf{a}\neq 0$ Condition (5.11) means that $f_j(z,s)=f_j(z,0)+a_jU(z,s)$.

For fixed σ the vector fields $\mathbf{L}_1, ..., \mathbf{L}_n$ define a CR structure of hypersurface type on $\mathbb{R}^{2n+1}$ which is integrable (or "realizable") since it admits the first integrals $z_1, ..., z_n$ and $w=s+\sigma\mathbf{a}\cdot z+i\varphi(z)$. The symbol of $\mathbf{L}_k$ is $\sigma(\mathbf{L}_k)=\frac{1}{2}\hat{z}_k-a_k\sigma\hat{s}-ib_k(z)\hat{s}$; the characteristic set of Σ is defined by the equations

$$\begin{aligned}&\hat{x}_k=2(\mathscr{Re}(a_k\sigma)+\mathscr{Im}\,b_k(z))\hat{s},\ \hat{y}_k=2(\mathscr{Im}(a_k\sigma)-\mathscr{Re}\,b_k(z))\hat{s},\\ &\hat{\xi}_i=\hat{\eta}_i=\hat{\sigma}=0,\qquad 1\leqslant i,k\leqslant n.\end{aligned}\tag{5.12}$$

We can parametrize the fibres of $\mathscr{C}har\,\Sigma$ by means of $\hat{s}$ and thus identify each of them to $\mathbb{R}$. The zero-section of $\mathscr{C}har\,\Sigma$ is defined by $\hat{s}=0$; it has no relevance to the hypocomplexity of the fibre-holomorphic Hamiltonian structure.

Here the $(n+1)\times n$ matrix F in (5.4) is equal to $\hat{s}[0\cdots 0\,\mathbf{a}]$ (each entry stands for a "column" $\mathbf{n}$-vector) and thus $F\neq 0$ on $\mathscr{C}har\,\Sigma\backslash 0$ provided $\mathbf{a}\neq 0$. The matrix G in (5.4) is the hermitian Hessian $\partial\bar{\partial}\varphi$ of φ, since $(2i)^{-1}[\mathbf{L}_k, \bar{\mathbf{L}}_l]=\partial^2\varphi/\partial\bar{z}_k\,\partial z_l$. If $[{}^{\mathbf{u}}_{\mathbf{v}}]$ is an eigenvector of the Levi matrix $\mathscr{L}$ for an eigenvalue $\lambda\neq 0$ and if $\mathbf{u}=(u_1, ..., u_n, u_{n+1})\in\mathbb{C}^{n+1}$ we have

$$u_i=0\ (i=1, ..., n),\qquad \hat{s}\bar{\mathbf{a}}\cdot\mathbf{v}=\lambda u_{n+1},\qquad \hat{s}u_{n+1}\mathbf{a}+G\mathbf{v}=\lambda\mathbf{v}. \tag{5.13}$$

Thus, if $\mathbf{v}$ is an eigenvector of G for λ and if $\bar{\mathbf{a}}\cdot\mathbf{v}=0$, then $[{}^{0}_{\mathbf{v}}]$ is an eigenvector of $\mathscr{L}$ for λ.

We shall submit $\mathbf{a}$ to the following requirement:

$$\mathbf{a}\text{ is an eigenvector of }G\text{ for its lowest eigenvalue }\lambda_0. \tag{5.14}$$

We can find an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of G and including $\mathbf{a}$. From what was just said it follows that every eigenvalue $\lambda>\lambda_0$ of G that has multiplicity $\mu\geqslant 1$ will also be an eigenvalue of $\mathscr{L}$ with multiplicity $\geqslant\mu$; if λ_0 has multiplicity $\mu_0>1$ relative to G, it also will be an eigenvalue of $\mathscr{L}$, with multiplicity $\geqslant\mu_0-1$. On the other hand, set

$$\begin{aligned}&\lambda_1=\tfrac{1}{2}\lambda_0-(\tfrac{1}{4}\lambda_0^2+\hat{s}^2\,\|\mathbf{a}\|^2)^{1/2},\\ &u_i=0\ (1\leqslant i\leqslant n),\qquad u_{n+1}=\hat{s}\,\|\mathbf{a}\|^2/\lambda_1,\qquad \mathbf{v}=\mathbf{a}.\end{aligned}$$

The definition of λ_1 entails $\lambda_1 = \lambda_0 + \hat{s}^2 \|\mathbf{a}\|^2/\lambda_1 = \lambda_0 + \hat{s}u_{n+1}$, hence $\lambda_1 \mathbf{a} = \lambda_0 \mathbf{a} + \hat{s}u_{n+1}\mathbf{a} = G\mathbf{a} + F\mathbf{u}$, which, together with the definition of $\mathbf{u}$, shows that Eqs. (5.13) are satisfied. We reach the following conclusion: if $\partial\bar{\partial}\varphi(0)$ is positive-semidefinite, $\lambda_1 < \lambda_0$ is an eigenvalue <0 of the Levi matrix $\mathscr{L}$ of Σ; in any case, all the eigenvalues of $\partial\bar{\partial}\varphi(0)$ strictly larger than λ_0 are also eigenvalues of $\mathscr{L}$.

Suppose that $\partial\bar{\partial}\varphi(0)$ has at least three eigenvalues >0 and at least three eigenvalues <0; the same will be true of $\mathscr{L}$ (at characteristic points lying above the submanifold $z=0$). In this case, according to results of [2], local integrability of the fibre-holomorphic Hamiltonian system will hold, and Theorem 5.1 will apply. We shall be able to conclude that the system of nonlinear equations (5.7) is stably solvable at every point of Σ lying on the submanifold $z=0$.

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